Spring School - April 2016 - Spartan/Macsenet
Francis Bach
Slides generously provided by Mark Schmidt

Modern Convex Optimization Methods for Large-Scale Empirical Risk Minimization (Part I: Primal Methods)

International Conference on Machine Learning

Peter Richtárik and Mark Schmidt

July 2015

Further reading:

-Dimitri Bertsekas. Convex Optimization Algorithms, Athena Scientific, 2015.
-Yurii Nesterov. Introductory lectures on convex optimization: a basic course. Kluwer Academic Publishers, 2004.
-Sebastien Bubeck. Convex optimization: Algorithms and complexity. Foundations and Trends in Machine Learning, 8(3-4):231–357, 2015.

Context: Big Data and Big Models

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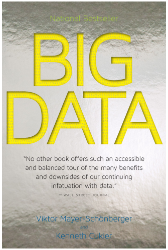

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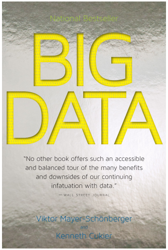

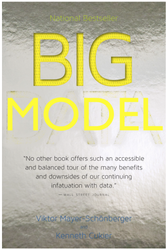

- Machine learning can use big data to fit richer models:
  - Bioinformatics.
  - Computer vision.
  - Speech recognition.
  - Product recommendation.
  - Machine translation.

Common Framework: Empirical Risk Minimization

- The most common framework is empirical risk minimization:

$$\min_{x \in \mathbb{R}^P} \frac{1}{N} \sum_{i=1}^{N} L(x, a_i, b_i) \quad + \quad \lambda r(x)$$

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  but also conditional random fields and deep neural networks.
- Main practical challenges:
  - Designing/learning good features a_i.
  - Efficiently solving the problem when N or P are very large.

Motivation: Why Learn about Convex Optimization?

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  - Tools from convex analysis are being extended to non-convex.

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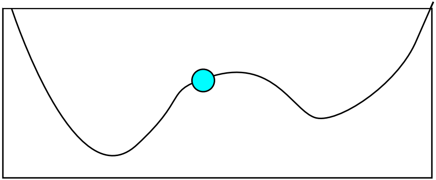

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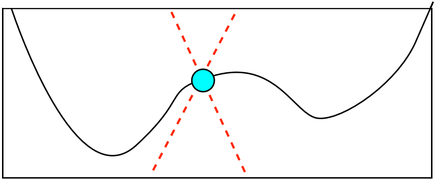

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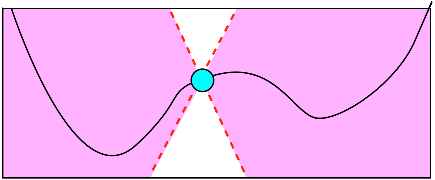

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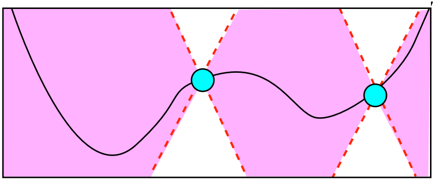

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- Optimization is hard, but assumptions make a big difference.

  (we went from impossible to very slow)

Convex Functions: Three Characterizations

A function f is convex if for all x and y we have

$$f(\theta x + (1-\theta)y) \leq \theta f(x) + (1-\theta)f(y), \quad for\ \theta \in [0,1].$$

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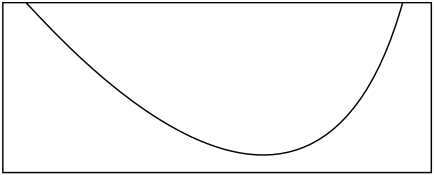

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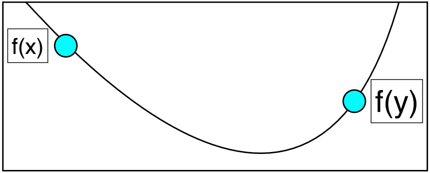

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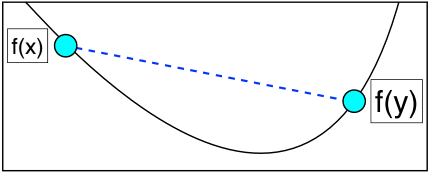

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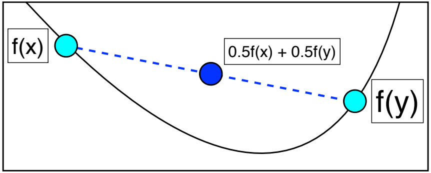

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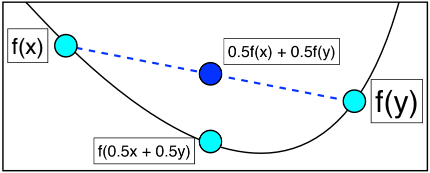

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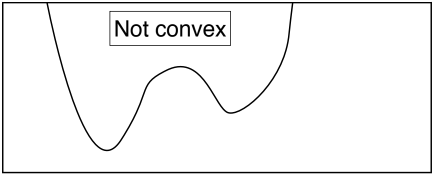

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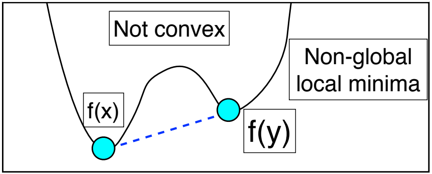

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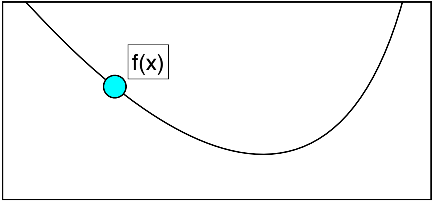

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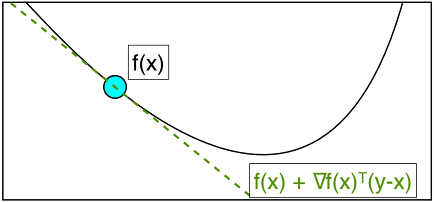

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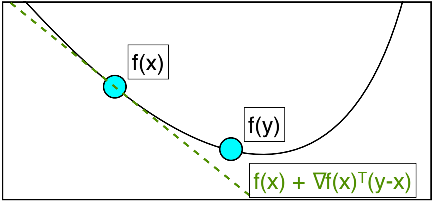

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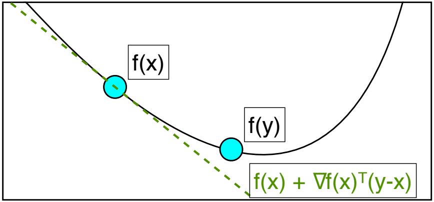

- If $\nabla f(y) = 0$, implies y is a a global minimizer.

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- The function is *flat or curved upwards* in every direction.
- This is usually the easiest way to show a function is convex.

Examples of Convex Functions

Some simple convex functions:

- $f(x) = c$
- $f(x) = a^T x$
- $f(x) = ax^2 + b$ (for $a > 0$)
- $f(x) = \exp(ax)$
- $f(x) = x \log x$ (for $x > 0$)
- $f(x) = \|x\|^2$
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Some other notable examples:

- $f(x, y) = \log(e^x + e^y)$
- $f(X) = \log \det X$ (for X positive-definite).
- $f(x, Y) = x^T Y^{-1} x$ (for Y positive-definite)

Operations that Preserve Convexity

1. Non-negative weighted sum:

$$f(x) = \theta_1 f_1(x) + \theta_2 f_2(x).$$

2. Composition with affine mapping:

$$g(x) = f(Ax + b).$$

3. Pointwise maximum:

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We know that $\| \cdot \|_p$ is a norm, so it follows from (2).

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The first term has Hessian $I \succ 0$, for the second term use (3) on the two (convex) arguments, then use (1) to put it all together.

Outline

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$$x^{t+1} = x^t - \alpha_t \nabla f(x^t).$$

  - Only have $O(P)$ iteration cost!
  - But how many iterations are needed?

Logistic Regression with 2-Norm Regularization

- Let's consider logistic regression with 2-norm regularization:

$$f(x) = \sum_{i=1}^{n} \log(1 + exp(-b_i(x^T a_i))) + \frac{\lambda}{2}\|x\|^2.$$

- Objective f is convex.
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- But we have

$$\mu I \preceq \nabla^2 f(x) \preceq LI,$$

for some L and μ.

$(L \leq \frac{1}{4}\|A\|_2^2 + \lambda,\ \mu \geq \lambda)$

Logistic Regression with 2-Norm Regularization

- Let's consider logistic regression with 2-norm regularization:

$$f(x) = \sum_{i=1}^{n} \log(1 + exp(-b_i(x^T a_i))) + \frac{\lambda}{2}\|x\|^2.$$

- Objective f is convex.
- First term is Lipschitz continuous, second term is not.
- But we have

$$\mu I \preceq \nabla^2 f(x) \preceq LI,$$

for some L and μ.

$(L \leq \frac{1}{4}\|A\|_2^2 + \lambda,\ \mu \geq \lambda)$

- We say that the gradient is Lipschitz-continuous.
- We say that the function is strongly-convex.

Properties of Lipschitz-Continuous Gradient

- From Taylor's theorem, for some z we have:

$$f(y) = f(x) + \nabla f(x)^T (y - x) + \frac{1}{2}(y - x)^T \nabla^2 f(z)(y - x)$$

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- Variant of gradient method if we set x^{t+1} to minimum y value:

$$x^{t+1} = x^t - \frac{1}{L}\nabla f(x^t).$$

- Plugging this value in:

$$f(x^{t+1}) \leq f(x^t) - \frac{1}{2L}\|\nabla f(x^t)\|^2.$$

- Guaranteed decrease of objective.

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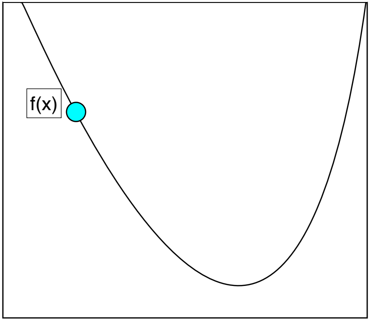

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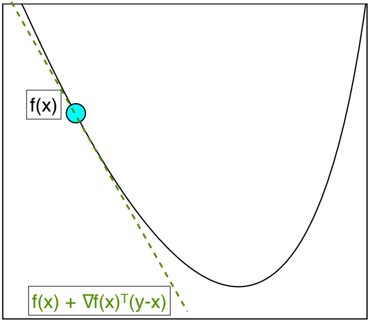

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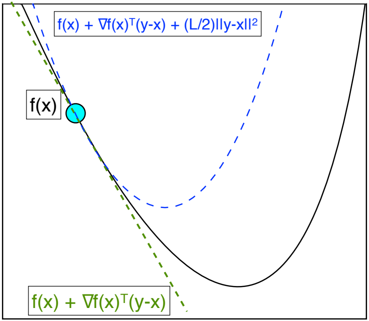

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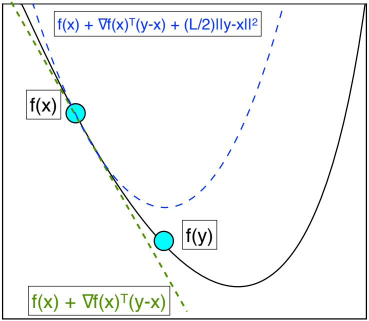

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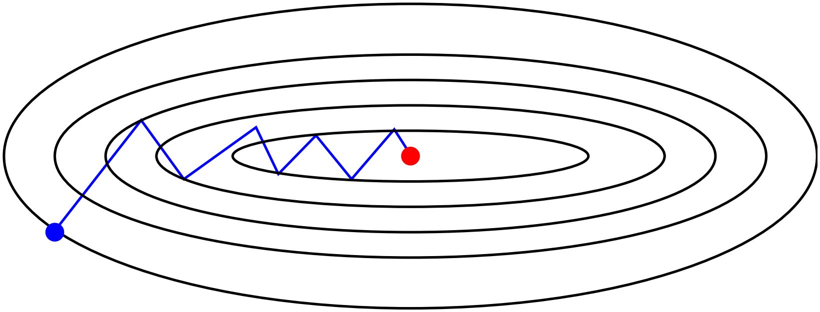

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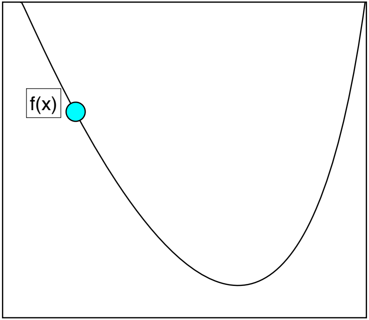

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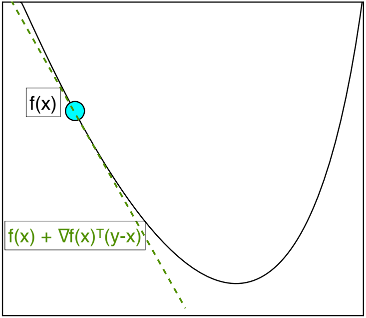

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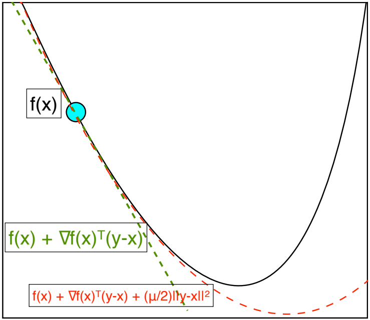

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- Minimize both sides in terms of y:

$$f(x^*) \geq f(x) - \frac{1}{2\mu}\|\nabla f(x)\|^2.$$

- Upper bound on how far we are from the solution.

Linear Convergence of Gradient Descent

- We have bounds on x^{t+1} and x^*:

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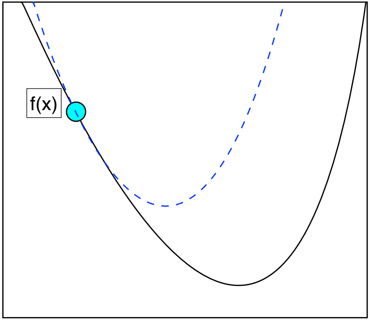

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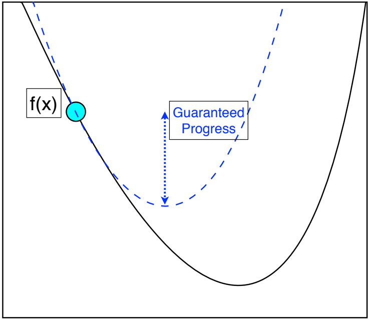

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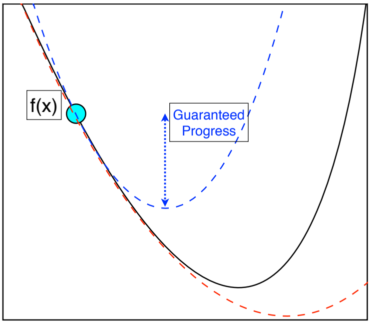

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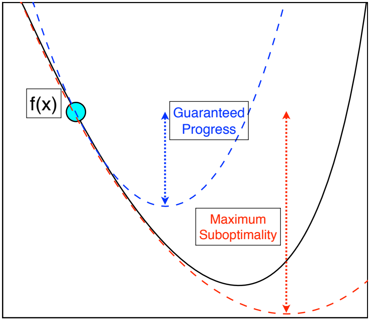

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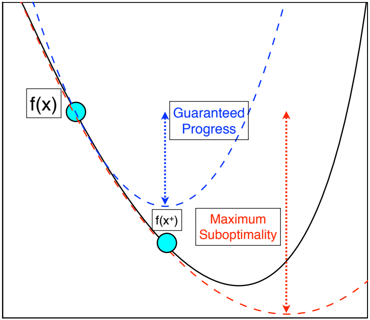

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- This gives a linear convergence rate:

$$f(x^t) - f(x^*) \leq \left(1 - \frac{\mu}{L}\right)^t [f(x^0) - f(x^*)]$$

- Each iteration multiplies the error by a fixed amount.

(very fast if μ/L is not too close to one)

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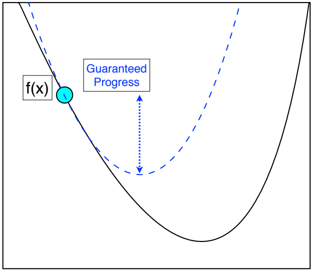

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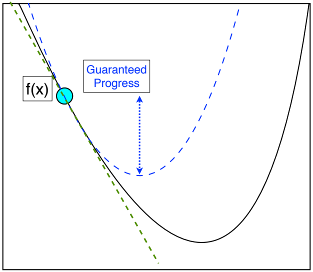

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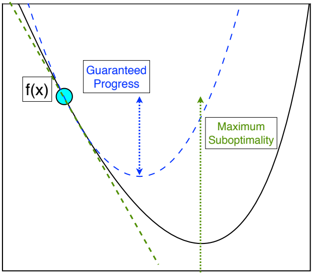

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- If f is convex, then $f + \lambda \|x\|^2$ is strongly-convex.

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- Also, check your derivative code!

$$\nabla_i f(x) \approx \frac{f(x + \delta e_i) - f(x)}{\delta}$$

- For large-scale problems you can check a random direction d:

$$\nabla f(x)^T d \approx \frac{f(x + \delta d) - f(x)}{\delta}$$

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- Rate is nearly-optimal for dimension-independent algorithm.
- Similar to heavy-ball/momentum and conjugate gradient.
- For logistic regression and many other losses, we can get linear convergence without strong-convexity [Luo & Tseng, 1993].

Newton's Method

- The oldest differentiable optimization method is Newton's.

  (also called IRLS for functions of the form $f(Ax)$)

- Modern form uses the update

$$x^{t+1} = x^t - \alpha d,$$

where d is a solution to the system

$$\nabla^2 f(x) d = \nabla f(x).$$

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- We can generalize the Armijo condition to

$$f(x^{t+1}) \leq f(x^t) + \gamma \alpha \nabla f(x^t)^T d.$$

- Has a natural step length of $\alpha = 1$.

  (always accepted when close to a minimizer)

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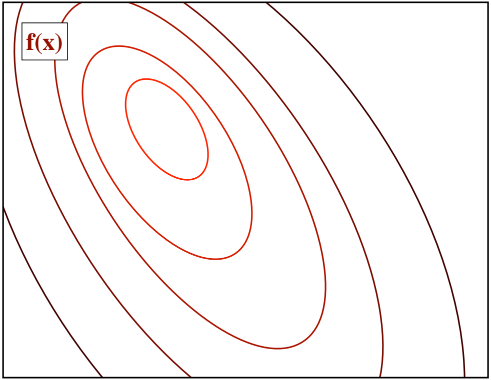

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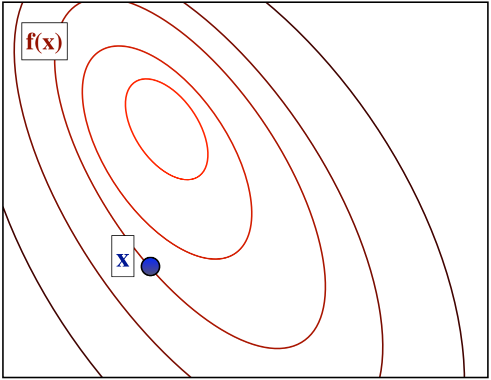

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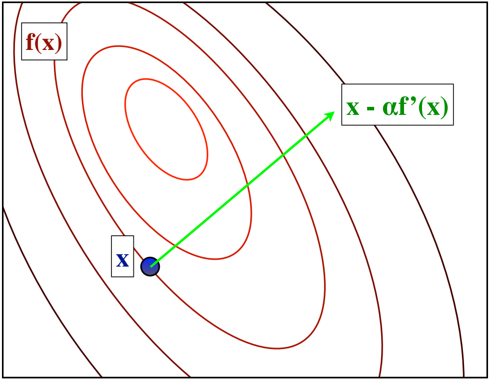

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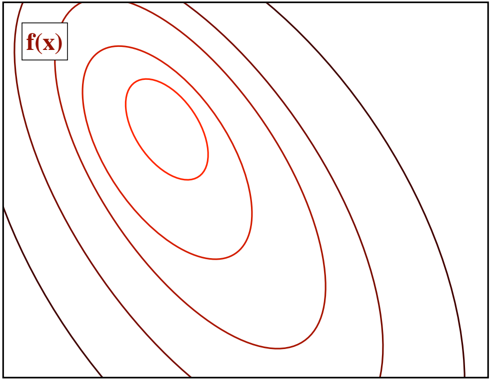

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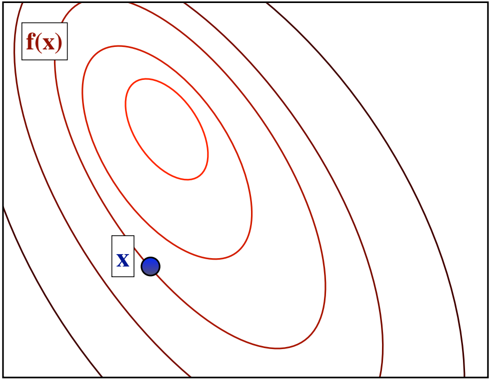

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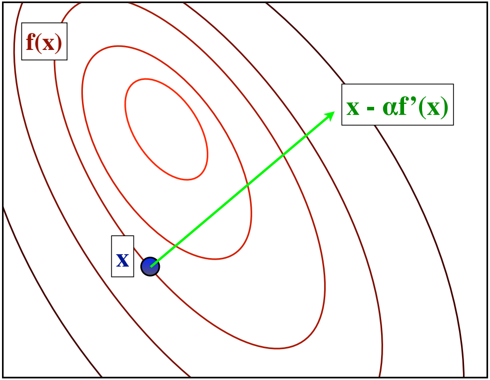

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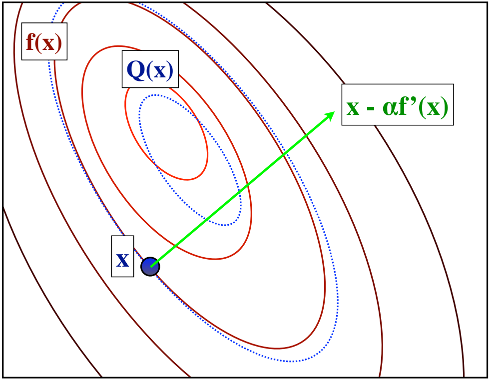

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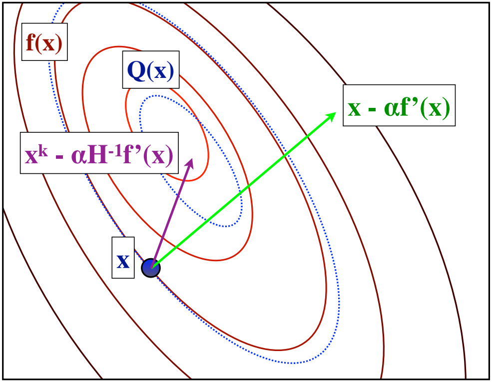

Convergence Rate of Newton's Method

- If $\nabla^2 f(x)$ is Lipschitz-continuous and $\nabla^2 f(x) \succeq \mu$, then close to x^* Newton's method has local superlinear convergence:

$$f(x^{t+1}) - f(x^*) \leq \rho_t[f(x^t) - f(x^*)],$$

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- But requires solving $\nabla^2 f(x) d = \nabla f(x)$.
- Get global rates under various assumptions (cubic-regularization/accelerated/self-concordant).

Newton's Method: Practical Issues

There are many practical variants of Newton's method:

- Modify the Hessian to be positive-definite.
- Only compute the Hessian every m iterations.
- Only use the diagonals of the Hessian.
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- Hessian-free: Compute d inexactly using Hessian-vector products:

$$\nabla^2 f(x)d = \lim_{\delta \to 0} \frac{\nabla f(x + \delta d) - \nabla f(x)}{\delta}$$

- Barzilai-Borwein: Choose a step-size that acts like the Hessian over the last iteration:

$$\alpha = \frac{(x^{t+1} - x^t)^T(\nabla f(x^{t+1}) - \nabla f(x^t))}{\|\nabla f(x^{t+1}) - f(x^t)\|^2}$$

Another related method is nonlinear conjugate gradient.

Outline

Big-N Problems

- Recall the regularized empirical risk minimization problem:

$$\min_{x \in \mathbb{R}^P} \frac{1}{N} \sum_{i=1}^{N} L(x, a_i, b_i) \quad + \quad \lambda r(x)$$

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- What if number of training examples N is very large?
  - E.g., ImageNet has more than 14 million annotated images.

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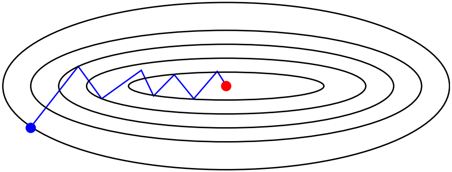

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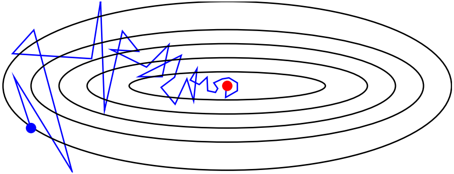

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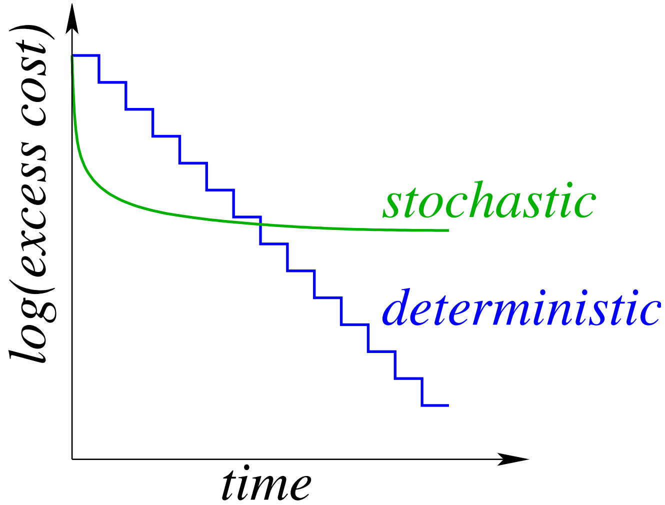
Convex	$O(1/t^2)$	$O(1/\sqrt{t})$
Strongly	$O((1-\sqrt{\mu/L})^t)$	$O(1/t)$

- Stochastic has low iteration cost but slow convergence rate.
    - Sublinear rate even in strongly-convex case.
    - Bounds are unimprovable if only unbiased gradient available.

Stochastic vs. Deterministic Convergence Rates

Plot of convergence rates in strongly-convex case:



Stochastic will be superior for low-accuracy/time situations.

Stochastic vs. Deterministic for Non-Smooth

- Consider the binary support vector machine objective:

$$f(x) = \sum_{i=1}^{n} \max\{0, 1 - b_i(x^T a_i)\} + \frac{\lambda}{2}\|x\|^2.$$

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- Other black-box methods (cutting plane) are not faster.
- For non-smooth problems:
  - Stochastic methods have same rate as smooth case.
  - Deterministic methods are not faster than stochastic method.
  - So use stochastic subgradient (iterations are n times faster).

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- At differentiable x:
  - Only subgradient is $\nabla f(x)$.
- At non-differentiable x:
  - We have a set of subgradients.
  - Called the sub-differential, $\partial f(x)$.
- Note that $0 \in \partial f(x)$ iff x is a global minimum.

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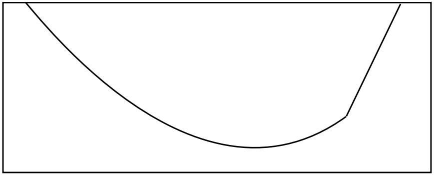

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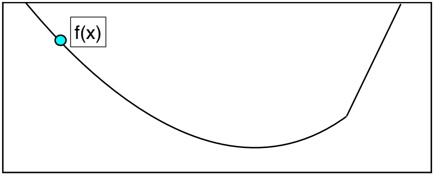

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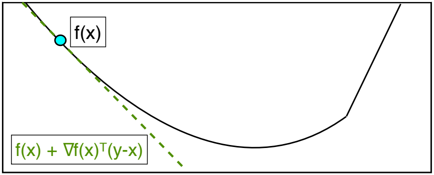

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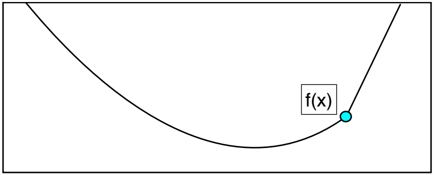

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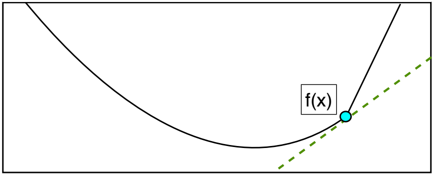

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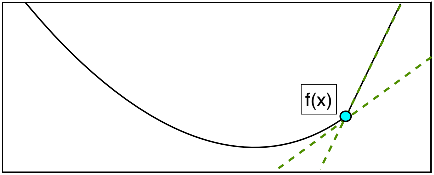

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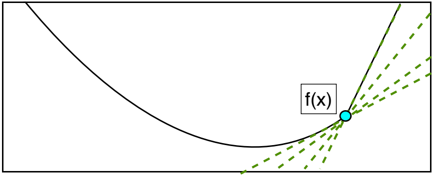

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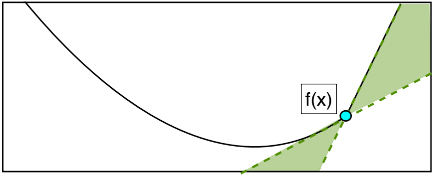

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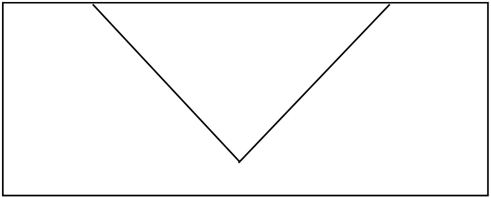

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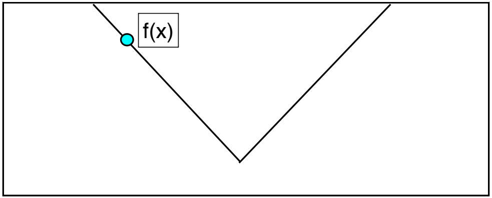

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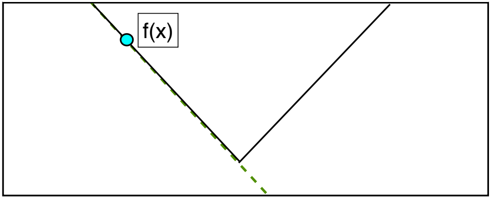

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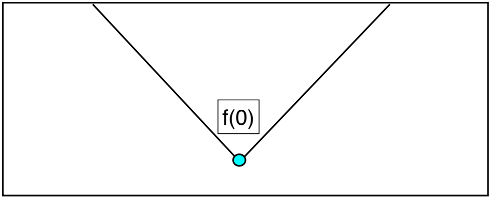

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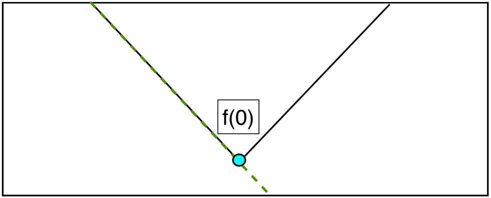

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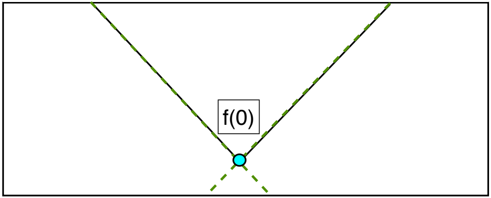

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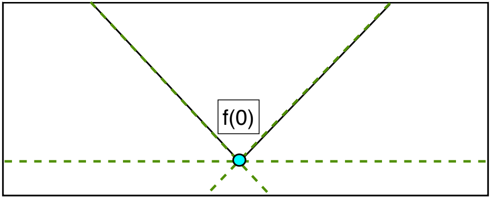

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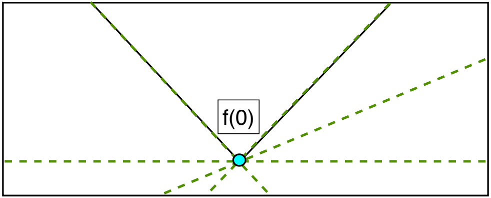

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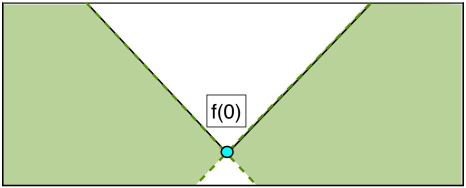

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(any convex combination of the gradients of the argmax)

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$$x^{t+1} = x^t - \alpha d,$$

  for some $d \in \partial f_i(x^t)$ for some random $i \in \{1, 2, \ldots, N\}$.

Stochastic Subgradient Methods in Practice

- The theory says to use decreasing sequence $\alpha_t = 1/\mu t$:

$$i_t = \mathrm{rand}(1, 2, \ldots, N), \quad \alpha_t = \frac{1}{\mu t}$$

$$x^{t+1} = x^t - \alpha_t f'_{i_t}(x^t).$$

  - $O(1/t)$ for smooth objectives.
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  - Later steps are tiny: $1/t$ gets small very quickly.
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  - Convergence rate is not robust to mis-specification of μ.
  - No adaptation to 'easier' problems than worst case.
- Tricks that can improve theoretical and practical properties:
  1. Use smaller initial step-sizes, that go to zero more slowly.
  2. Take a (weighted) average of the iterations or gradients:

$$\bar{x}_t = \sum_{i=1}^{t} \omega_t x_t, \quad \bar{d}_t = \sum_{i=1}^{t} \delta_t d_t.$$

Speeding up Stochastic Subgradient Methods

Works that support using large steps and averaging:

- Rakhlin et at. [2011]:
  - Averaging later iterations achieves $O(1/t)$ in non-smooth case.
- Nesterov [2007], Xiao [2010]:
  - Gradient averaging improves constants ('dual averaging').
  - Finds non-zero variables with sparse regularizers.
- Bach & Moulines [2011]:
  - $\alpha_t = O(1/t^\beta)$ for $\beta \in (0.5, 1)$ more robust than $\alpha_t = O(1/t)$.

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- Nedic & Bertsekas [2000]:
  - Constant step size ($\alpha_t = \alpha$) achieves rate of

$$\mathbb{E}[f(x^t)] - f(x^*) \leq (1 - 2\mu\alpha)^t (f(x^0) - f(x^*)) + O(\alpha).$$

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- Polyak & Juditsky [1992]:
  - In smooth case, iterate averaging is asymptotically optimal.
  - Achieves same rate as optimal stochastic Newton method.

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Stochastic Newton Methods?

- Should we use accelerated/Newton-like stochastic methods?
  - These do not improve the convergence rate.
- But some positive results exist.
  - Ghadimi & Lan [2010]:
    - Acceleration can improve dependence on L and μ.
    - Improves performance at start or if noise is small.
  - Duchi et al. [2010]:
    - Newton-like methods can improve regret bounds.
  - Bach & Moulines [2013]:
    - Newton-like method achieves $O(1/t)$ without strong-convexity.
      (under extra self-concordance assumption)

Outline

Big-N Problems

- Recall the regularized empirical risk minimization problem:

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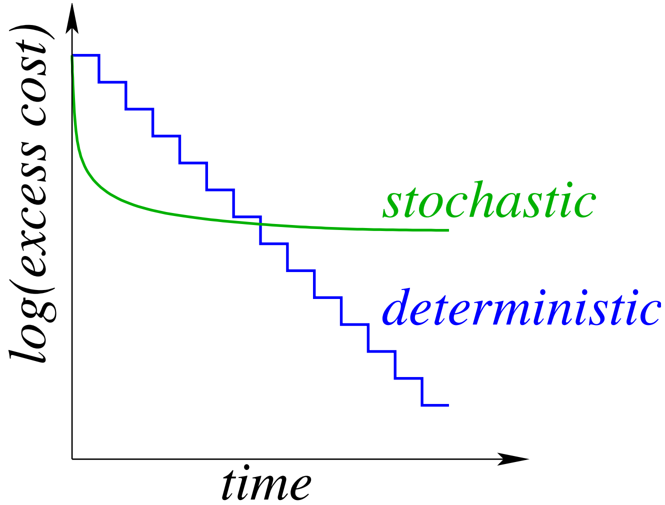
- For minimizing finite sums, can we design a better method?

Motivation for Hybrid Methods



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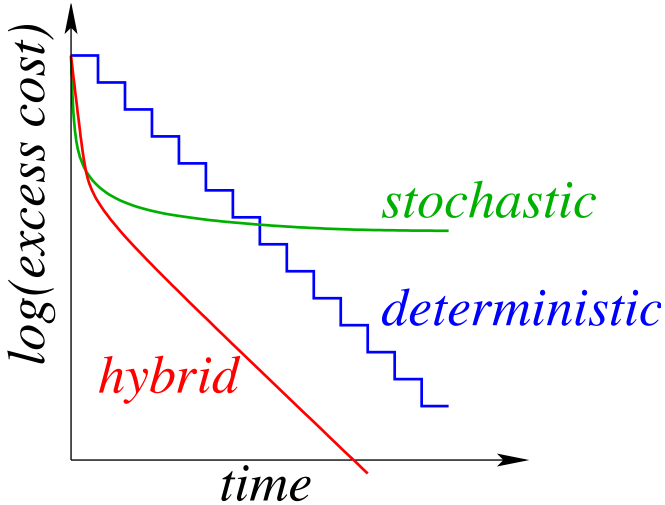

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- A common variant is to use larger sample $\mathcal{B}^t$,

$$\frac{1}{|\mathcal{B}^t|}\sum_{i \in \mathcal{B}^t} f_i'(x^t) \approx \frac{1}{N}\sum_{i=1}^{N} f_i(x^t).$$

Approach 1: Batching

- The SG method with a sample $\mathcal{B}^t$ uses iterations

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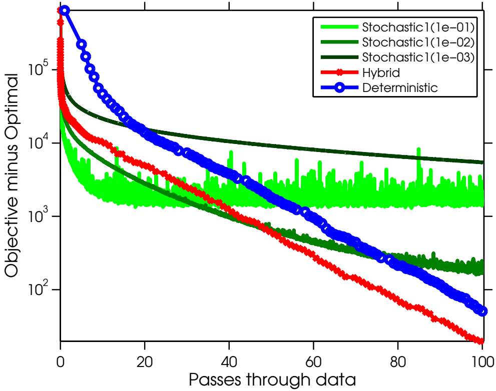
  [Bertsekas & Tsitsiklis, 1996]
- We can choose $|\mathcal{B}^t|$ to achieve a linear convergence rate:
  - Early iterations are cheap like SG iterations.
  - Later iterations can use a Newton-like method.

Evaluation on Chain-Structured CRFs

Results on chain-structured conditional random field:



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  - Assumes gradients of non-selected examples don't change.
  - Assumption becomes accurate as $||x^{t+1} - x^t|| \to 0$.

Convergence Rate of SAG

- If each f_i' is L−continuous and f is strongly-convex, with $\alpha_t = 1/16L$ SAG has

$$\mathbb{E}[f(x^t) - f(x^*)] \leqslant \left(1 - \min\left\{\frac{\mu}{16L}, \frac{1}{8N}\right\}\right)^t C,$$

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- Linear convergence rate but only 1 gradient per iteration.
    - For well-conditioned problems, constant reduction per pass:

$$\left(1 - \frac{1}{8N}\right)^N \leq \exp\left(-\frac{1}{8}\right) = 0.8825.$$

    - For ill-conditioned problems, almost same as deterministic method (but N times faster).

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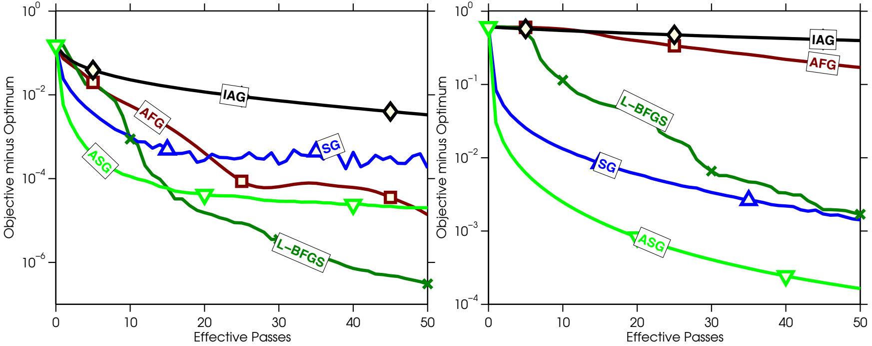
  - SAG: $O(\max\{N, \frac{L}{\mu}\}\log(1/\epsilon))$.

Comparing Deterministic and Stochatic Methods

- quantum ($n = 50000$, $p = 78$) and rcv1 ($n = 697641$, $p = 47236$)



SAG Compared to FG and SG Methods

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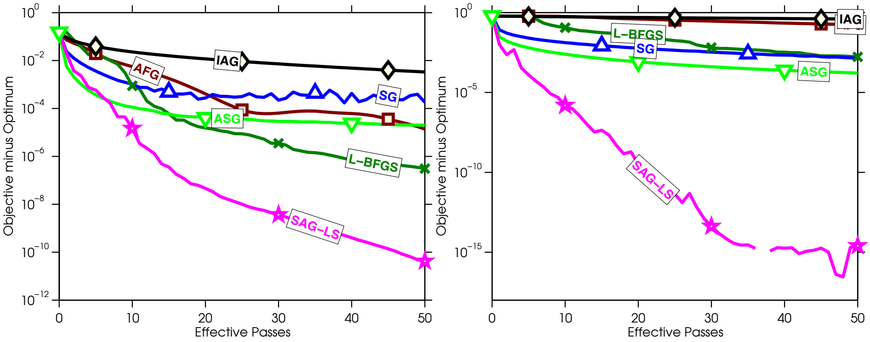

Other Linearly-Convergent Stochastic Methods

- Newer stochastic algorithms are now available with linear rates:
  - Stochastic dual coordinate ascent [Shalev-Schwartz & Zhang, 2013]
  - Incremental surrogate optimization [Mairal, 2013].
  - Stochastic variance-reduced gradient (SVRG) [Johnson & Zhang, 2013, Konecny & Richtarik, 2013, Mahdavi et al., 2013, Zhang et al., 2013]
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  - SAGA [Defazio et al., 2014]
- SVRG has a much lower memory requirement (later in talk).
- There are also non-smooth extensions (last part of talk).

SAG Implementation Issues

- Basic SAG algorithm:
  - while(1)
  - Sample i from $\{1, 2, \ldots, N\}$.
  - Compute $f_i'(x)$.
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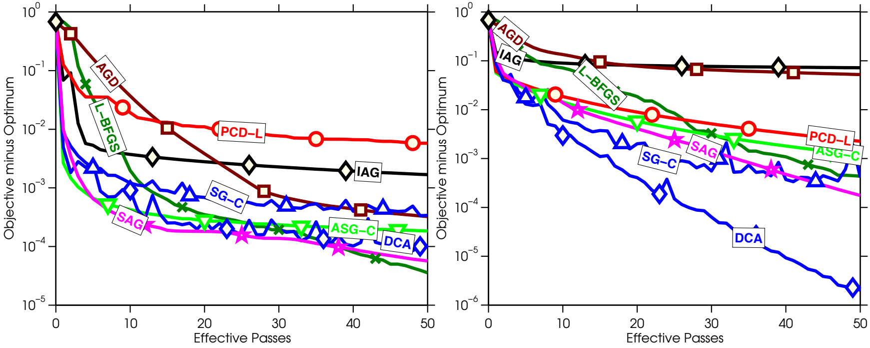
  - Termination criterion.
  - Acceleration [Lin et al., 2015].
  - Adaptive non-uniform sampling [Schmidt et al., 2013]:
    - Sample gradients that change quickly more often.

SAG with Adaptive Non-Uniform Sampling

- protein ($n = 145751$, $p = 74$) and sido ($n = 12678$, $p = 4932$)



- Datasets where SAG had the worst relative performance.

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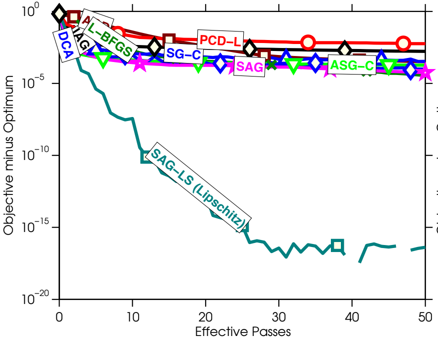

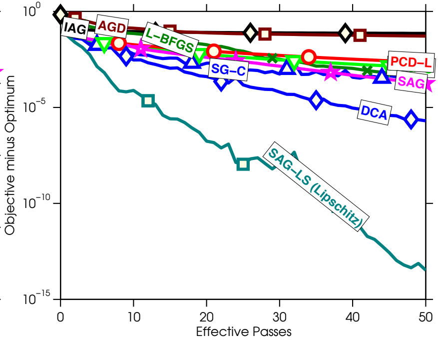

- Lipschitz sampling helps a lot.

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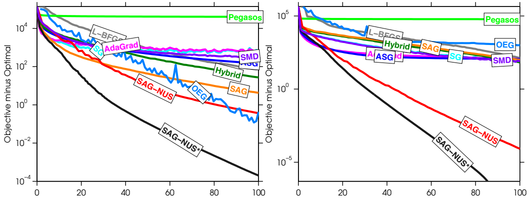

(optical character and named-entity recognition tasks)

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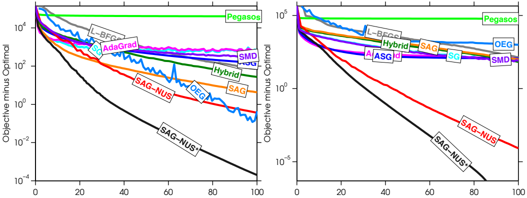

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- If the above don't work, use SVRG...

Stochastic Variance-Reduced Gradient

SVRG algorithm:

- Start with x_0
- for $s = 0, 1, 2 \dots$
  - $d_s = \frac{1}{N} \sum_{i=1}^{N} f_i'(x_s)$
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  - for $t = 1, 2, \dots m$
    - Randomly pick $i_t \in \{1, 2, \dots, N\}$
    - $x^t = x^{t-1} - \alpha_t(f_{i_t}'(x^{t-1}) - f_{i_t}'(x_s) + d_s)$.
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Requires 2 gradients per iteration and occasional full passes, but only requires storing d_s and x_s.

Outline

Motivation: Sparse Regularization

- Recall the regularized empirical risk minimization problem:

$$\min_{x\in\mathbb{R}^P} \frac{1}{N}\sum_{i=1}^{N} L(x, a_i, b_i) \quad + \quad \lambda r(x)$$

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- Faster methods for specific non-smooth problems?

Smoothing Approximations of Non-Smooth Functions

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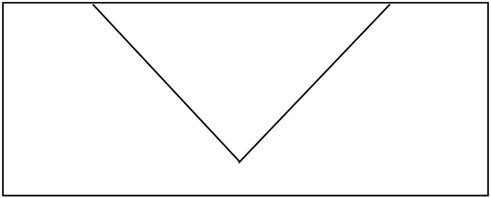

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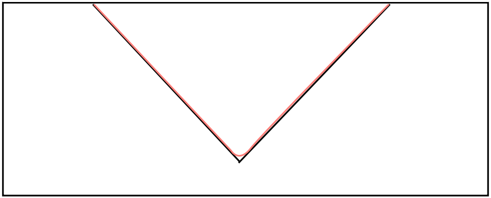

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- Generic smoothing strategy: strongly-convex regularization of convex conjugate.[Nesterov, 2005]

Discussion of Smoothing Approach

- Nesterov [2005] shows that:
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- In practice:
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  - Use faster algorithms like L-BFGS, SAG, or SVRG.
- You can get the $O(1/t)$ rate for $\min_x \max\{f_i(x)\}$ for f_i convex and smooth using *mirror-prox* method.[Nemirovski, 2004]
  - See also Chambolle & Pock [2010].

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- These are smooth objective with 'simple' constraints.

$$\min_{x\in\mathcal{C}} f(x).$$

Optimization with Simple Constraints

- Recall: gradient descent minimizes quadratic approximation:

$$x^{t+1} = \operatorname*{argmin}_y \left\{ f(x^t) + \nabla f(x^t)^T (y - x^t) + \frac{1}{2\alpha_t} \|y - x^t\|^2 \right\}.$$

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- Equivalent to projection of gradient descent:

$$x_t^{GD} = x^t - \alpha_t \nabla f(x^t),$$

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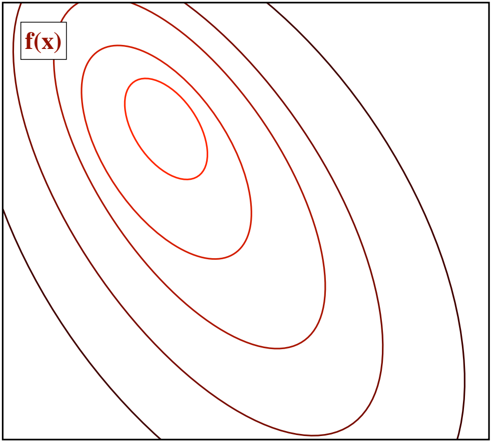
Gradient Projection



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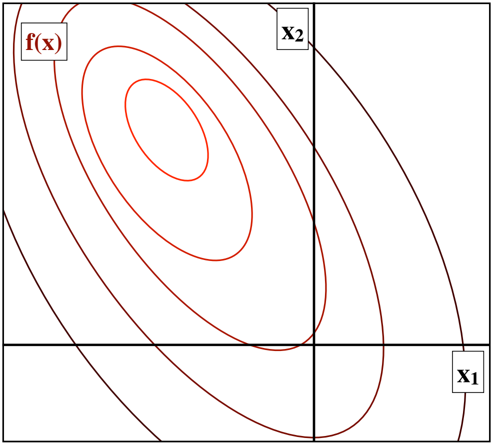

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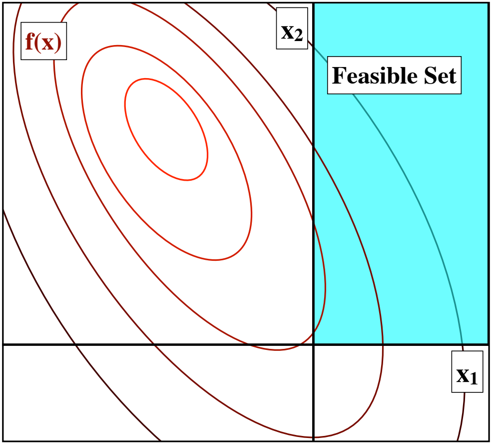

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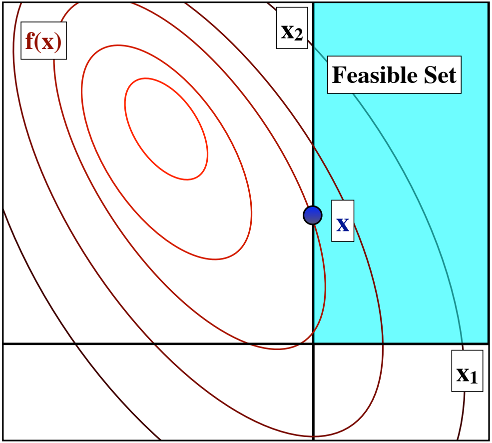

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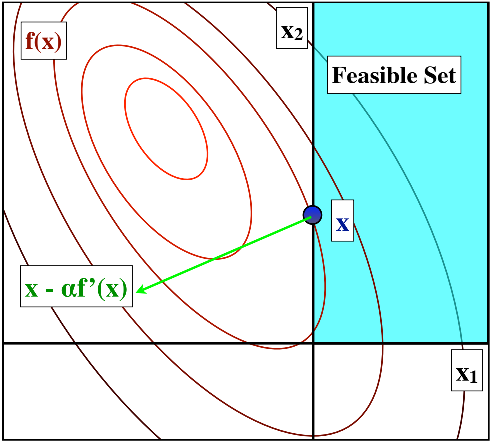

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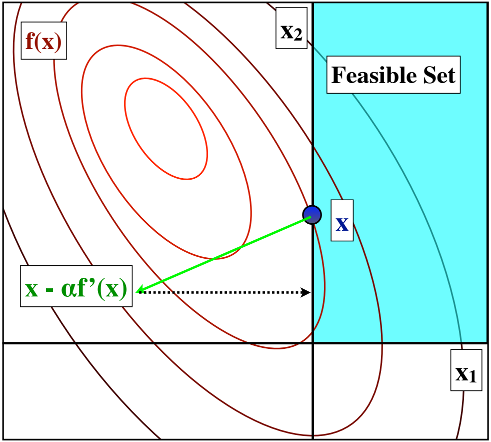

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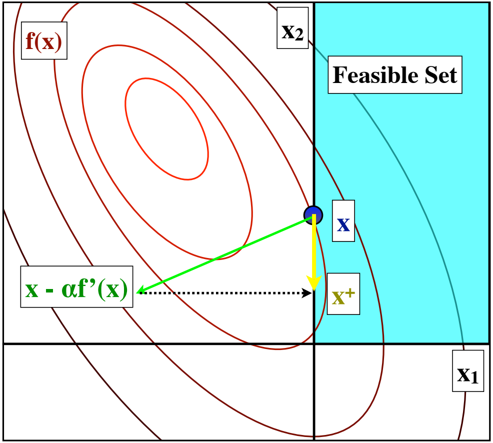

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- Can do many of the same tricks (i.e. line-search, acceleration, Barzilai-Borwein, SAG, SVRG).
- For projected Newton, you need to do an expensive projection under $\|\cdot\|_{H_t}$.
  - Two-metric projection methods allow Newton-like strategy for bound constraints.
  - Inexact Newton methods allow Newton-like like strategy for optimizing costly functions with simple constraints.

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- Intersection of simple sets: Dykstra's algorithm.

We can solve large instances of problems with these constraints.

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- Equivalent to using the approximation

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- Convergence rates are still the same as for minimizing f.

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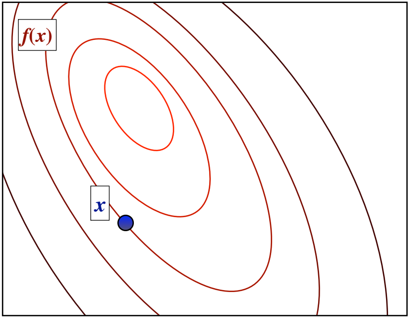

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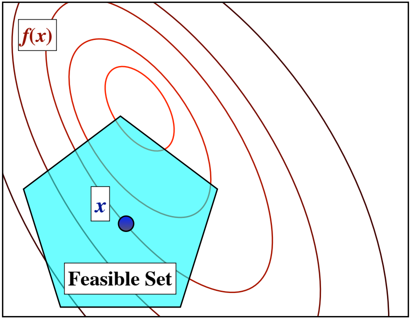

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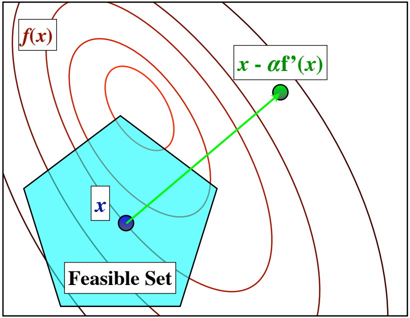

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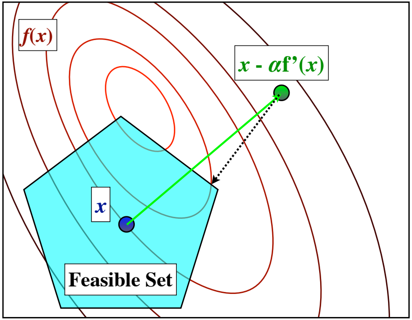

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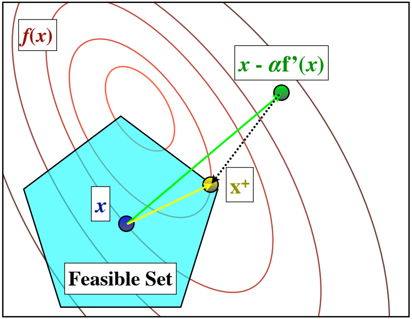

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- We can again do many of the same tricks (line-search, acceleration, Barzilai-Borwein, two-metric projection, inexact proximal operators, SAG, SVRG).

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- If prox can not be computed exactly: Linearized ADMM.

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- In some cases the projected gradient step

$$x^{t+1} = \underset{y \in \mathcal{C}}{\operatorname{argmin}} \left\{ f(x^t) + \nabla f(x^t)^T (y - x^t) + \frac{1}{2\alpha_t} \|y - x^t\|^2 \right\},$$

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- Iterate can be written as convex combination of vertices of $\mathcal{C}$.
- $O(1/t)$ rate for smooth convex objectives, some linear convergence results for smooth and strongly-convex.[Jaggi, 2013]

Alternatives to Quadratic/Linear Surrogates

- Mirror descent uses the iterations[Beck & Teboulle, 2003]

$$x^{t+1} = \operatorname*{argmin}_{y \in \mathcal{C}} \left\{ f(x) + \nabla f(x)^T (y - x^t) + \frac{1}{2\alpha_t} \mathcal{D}(x^t, y) \right\},$$

  where $\mathcal{D}$ is a Bregman-divergence:
  - $\mathcal{D} = \|x^t - y\|^2$ (gradient method).
  - $\mathcal{D} = \|x^t - y\|_H^2$ (Newton's method).
  - $\mathcal{D} = \sum_i x_i^t \log(\frac{x_i^t}{y_i}) - \sum_i (x_i^t - y_i)$ (exponentiated gradient).

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- Mairal [2013,2014] considers general surrogate optimization:

$$x^{t+1} = \operatorname*{argmin}_{y \in \mathcal{C}} \{g(y)\},$$

  where g upper bounds f, $g(x^t) = f(x^t)$, $\nabla g(x^t) = \nabla f(x^t)$, and $\nabla g - \nabla f$ is Lipschitz-continuous.
  - Get $O(1/k)$ and linear convergence rates depending on $g - f$.

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- SVM non-smooth strongly-convex primal:

$$\min_{x} C \sum_{i=1}^{N} \max\{0, 1 - b_i a_i^T x\} + \frac{1}{2}\|x\|^2.$$

- SVM smooth dual:

$$\min_{0 \leq \alpha \leq C} \frac{1}{2}\alpha^T A A^T \alpha - \sum_{i=1}^{N} \alpha_i$$

- Smooth bound constrained problem:
  - Two-metric projection (efficient Newton-liked method).
  - Randomized coordinate descent (part 2 of this talk).

Summary

Summary:

- Part 1: Convex functions have special properties that allow us to efficiently minimize them.
- Part 2: Gradient-based methods allow elegant scaling with dimensionality of problem.
- Part 3: Stochastic-gradient methods allow scaling with number of training examples, at cost of slower convergence rate.
- Part 4: For finite datasets, SAG fixes convergence rate of stochastic gradient methods, and SVRG fixes memory problem of SAG.
- Part 5: These building blocks can be extended to solve a variety of constrained and non-smooth problems.